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On interface crack in laminated anisotropic medium

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Abstract

In this paper, the interface crack problems of a multilayered anisotropic medium under a state of generalized plane deformation are considered within the framework of anisotropic theory. A general solution procedure is introduced such that it can be uniformly applied to media with transversely isotropic, orthotropic, monoclinic, etc. layers. The problem is reduced to the solution of a system of singular integral equations by means of Fourier transform method and the stiffness matrix formulation. A Jacobi polynomial technique is then used to solve the integral equations numerically. The stress intensity factors are provided. The stress intensity factors have been calculated numerically and displayed graphically.

1. Introduction

In recent years, the expanded use of composite materials in many modern engineering applications has attracted considerable research attention to the damage analysis of a laminated medium. In order to ensure structural integrity, understanding of fracture behavior of this structure is of great importance. Cracks are likely to occur on grain boundaries and bimaterial interfaces in polycrystal alloys and composite materials. It is, therefore, of practical importance to be able to assess the stress fields near such interface cracks.

The problem of an interface crack between dissimilar isotropic half-spaces was studied by many authors such as Williams (1959), England (1965), Erdogan (1965), etc. Solutions to interface crack problems of dissimilar anisotropic half-space can be found in the works of Clements (1971), Ting (1990) and Suo (1990). In Chatterjee (1977), the problems of laminated composite beam-type structures containing an interlaminar crack were considered where the interlaminar is isotropic. This work may have a significant contribution to the failure of laminated composites. Thangjitham et al. (1993) extended this method to the interlaminar crack problems of a laminated anisotropic

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medium, assuming that the crack was embedded within a thin isotropic matrix interlaminar region bounded by two adjacent anisotropic layers. This assumption circumvents the difficulties associated with the oscillatory singular behavior of the stress field near the crack tip, but raises questions, such as how to determine the thickness of the interlaminar; and it is also only valid for composite material laminates. When each layer of the laminate is homogeneous and anisotropic other than composite, i.e. without the same matrix, the thin interlaminar will not exist.

Many of the anisotropic materials used in engineering exhibit orthotropic and transverse isotropic symmetry; in such media degenerate cases may exist, i.e. an eigenvalue p_i may be a multiple root as obtained from Stroh's method. If the degeneracy is 'semisimple', i.e. if the number of independent eigenvectors is equal to the multiplicity of the eigenvalue, the solution can be obtained in the same way to the 'simple' (Ting, 1988). If the degeneracy is 'nonsemisimple', a different method should be used to solve the problem, which is the main purpose of this paper.

In this paper, we consider the interface crack problem of a laminated anisotropic plate under a state of generalized plane deformation. Instead of assuming a thin isotropic matrix interlaminar region, we assume that each layer of the laminate is ideally bonded. For the sake of generality, all layers are considered to be arbitrary anisotropic, which may be isotropic, orthotropic or transverse isotropic symmetry, etc. and they may be composite or homogeneous. The general solution procedure is introduced for these degenerates cases. As an efficient approach to the analysis of layered media, the stiffness matrix formulation (Chatterjee, 1977; Choi and Thangjitham, 1991) is extended to the current crack problem. By means of Fourier transforms the problem is reduced to a system of simultaneous algebraic equations by using the Jacobi polynomial approximation, the stress intensity factors are obtained in terms of the solutions of the corresponding integral equations. Numerical methods are employed to determine the stress intensity factors, which have been displayed graphically.

2. Governing equations and general solution

As shown in Fig. 1, consider an N-layer laminated anisotropic plate containing an interface crack of length $2a$ located between the j-th and $(j+1)$-th layers. The laminate layers are perfectly bonded except at the crack.

In order to maintain generality in the analysis, each layer of the laminated plate is considered to be arbitrary anisotropic. The surface tractions are applied such that all field variables are only functions of x_1 and x_2, i.e. a state of generalized plane deformation is assumed:

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = u_3(x_1, x_2)$$

where, u_1, u_2, u_3 represent the displacements in the x_1-, x_2-, x_3-directions, respectively. The general displacement vector is defined as

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$$

The transform pair for a generic function $g(x_1, x_2)$ is defined as

$$\tilde{g}(s, x_2) = \int_{-\infty}^{+\infty} g(x, x_2)\, \mathrm{e}^{isx}\, \mathrm{d}x, \quad g(x, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{g}(s, x_2)\, \mathrm{e}^{isx}\, \mathrm{d}s \tag{1}$$

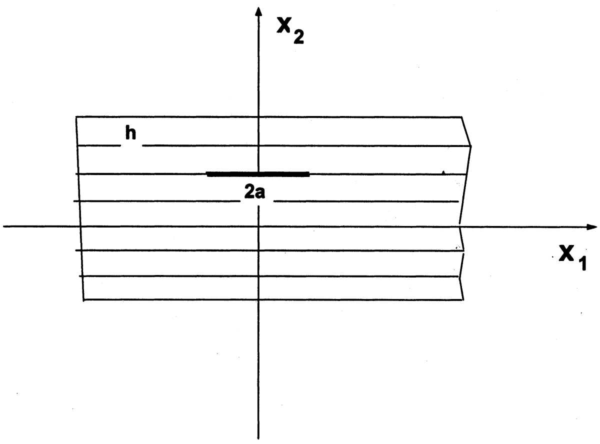

Fig. 1. Configuration of an anisotropic laminate with an interface crack.

where the overtilde $\sim$ denotes the transformed quantity, s is the transform variable, and we let $x_1 = x$.

Introduce the following matrices

$$(\mathbf{Q})_{ik} = C_{i1k1}, \quad (\mathbf{R})_{ik} = C_{i1k2}, \quad (\mathbf{T})_{ik} = C_{i2k2}, \quad (i, k = 1, 2, 3)$$

where C_{ijrs} are the elasticity constants. Matrices $\mathbf{Q}$ and $\mathbf{T}$ are symmetric and positive definite due to the strain energy being positive.

By letting $y = -isx_2$ and using the Fourier transform, the equations of equilibrium in the absence of body forces may be expressed as follows:

$$\mathbf{T}\frac{\partial^2 \tilde{\mathbf{u}}}{\partial y^2} + (\mathbf{R}+\mathbf{R}^T)\frac{\partial \tilde{\mathbf{u}}}{\partial y} + \mathbf{Q}\tilde{\mathbf{u}} = 0 \tag{2}$$

or in the form

$$\frac{\partial}{\partial y}\boldsymbol{\eta} = \mathbf{N}\boldsymbol{\eta} \tag{3}$$

where

$$\boldsymbol{\eta} = \begin{bmatrix} \tilde{\mathbf{u}} \\ \left[\mathbf{R}^T + \mathbf{T}\dfrac{\partial}{\partial \mathbf{y}}\right]\tilde{\mathbf{u}} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -\mathbf{T}^{-1}\mathbf{R}^T & \mathbf{T}^{-1} \\ \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T - \mathbf{Q} & -\mathbf{R}\mathbf{T}^{-1} \end{bmatrix} \tag{4}$$

It can be proved that $\mathbf{N}$ admits no real eigenvalues, and its six eigenvalues form three conjugate pairs (Suo, 1990). Without the loss of generality, let p_i $(i = 1, 2, 3)$ be the eigenvalues with positive imaginary part and $\boldsymbol{\xi}_i$ be the corresponding eigenvectors. In this paper, the assumption that p_i are distinct is not necessary.

If p_i, is a single root, the solutions to eqn (3) can be readily obtained as

$$\boldsymbol{\eta}_i(y, p_i) = \boldsymbol{\xi}_i\, e^{yp_i} \tag{5}$$

If p_i is a k-fold root ($1 \leqslant k \leqslant 3$) and there exists only one independent eigenvector ξ_i associated with p_i, the solution to eqn (3) should be written as

$$\boldsymbol{\eta}_i(y, p_i) = \boldsymbol{\xi}_i\, e^{yp_i}$$

$$\boldsymbol{\eta}_{i+j}(y, p_i) = \left[\boldsymbol{\xi}_{i+j} + \sum_{n=1}^{i} \frac{1}{n} y^n (\mathbf{N} - p_i \mathbf{I})^n \boldsymbol{\xi}_{i+j}\right] e^{yp_i} \quad (j = 1, \ldots, k-1) \tag{6}$$

where

$$(\mathbf{N} - p_i \mathbf{I})^{j+1} \boldsymbol{\xi}_{i+j} = 0, \quad \text{and} \quad (\mathbf{N} - p_i \mathbf{I})^{j} \boldsymbol{\xi}_{i+j} \neq 0$$

In fact

$$\boldsymbol{\xi}_{i+j} = \frac{d^j \boldsymbol{\xi}_i}{dp_i^j} = \boldsymbol{\xi}_i^{(j)} = j!(\mathbf{N} - p_i \mathbf{I})^{-j} \boldsymbol{\xi}_i$$

then

$$\boldsymbol{\eta}_{i+j} = \left[\boldsymbol{\xi}_i^{(j)} + \sum_{n=1}^{j} \frac{j!}{(j-n)!n} y^n \boldsymbol{\xi}_i^{(j-n)}\right] e^{yp_i} \tag{7}$$

So, the general solution to eqn (3) is

$$\boldsymbol{\eta} = \sum_{n=1}^{3} [\boldsymbol{\Xi}_i(s) \boldsymbol{\eta}_i(y, p_i) + \boldsymbol{\Delta}_i(s) \boldsymbol{\eta}_i(y, \bar{p}_i)]$$

$$= \sum_{i=1}^{3} [\boldsymbol{\Xi}_i(s) \boldsymbol{\zeta}_i(y)\, e^{yp_i} + \boldsymbol{\Delta}_i(s) \bar{\boldsymbol{\zeta}}_i(y)\, e^{y\bar{p}_i}] \tag{8}$$

where $\boldsymbol{\zeta}_i$ can be determined by eqns (6) and (7).

Let

$$\boldsymbol{\xi}_i = \begin{bmatrix} \mathbf{a}_i \\ \mathbf{b}_i \end{bmatrix} \quad \text{and} \quad \boldsymbol{\xi}_i' = \begin{bmatrix} \mathbf{a}_i' \\ \mathbf{b}_i' \end{bmatrix} \quad \ldots \quad \boldsymbol{\xi}_i^{(j)} = \begin{bmatrix} \mathbf{a}_i^{(j)} \\ \mathbf{b}_i^{(j)} \end{bmatrix}$$

$$\boldsymbol{\zeta}_i = \begin{bmatrix} \mathbf{g}_i(y) \\ \mathbf{h}_i(y) \end{bmatrix}$$

then, it follows that

$$\tilde{\mathbf{u}} = \sum_{i=1}^{3} [\boldsymbol{\Xi}_i(s) \mathbf{g}_i\, e^{yp_i} + \boldsymbol{\Delta}_i(s) \bar{\mathbf{g}}_i\, e^{y\bar{p}_i}] = \mathbf{A}\mathbf{E}_1 + \boldsymbol{\Xi}\bar{\mathbf{A}}\mathbf{E}_2\boldsymbol{\Delta} \tag{9}$$

in which

$$\mathbf{A}(y) = [\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3], \quad \mathbf{\Xi} = [\Xi_1 \quad \Xi_2 \quad \Xi_3]^T, \quad \mathbf{\Delta} = [\mathbf{\Delta}_1 \quad \mathbf{\Delta}_2 \quad \mathbf{\Delta}_3]^T$$

$$\mathbf{E}_1 = \operatorname{diag}[\mathrm{e}^{-isp_i x_2}], \quad \mathbf{E}_2 = \operatorname{diag}[\mathrm{e}^{-is\bar{p}_i x_2}] \quad (i = 1, 2, 3) \tag{10}$$

From the constitutive relations, the stresses are given by

$$\tilde{\mathbf{t}} = \begin{bmatrix} \tilde{\boldsymbol{\sigma}}_{12} \\ \tilde{\boldsymbol{\sigma}}_{22} \\ \boldsymbol{\sigma}_{32} \end{bmatrix} = \left(-is\mathbf{R}^T + \mathbf{T}\frac{\partial}{\partial x_2} \right) \tilde{\mathbf{u}} = -is\left(\mathbf{R}^T + \mathbf{T}\frac{\partial}{\partial y} \right) \tilde{\mathbf{u}} \tag{11}$$

Therefore, eqn (4) can be rewritten as

$$\boldsymbol{\eta} = \begin{bmatrix} \tilde{\mathbf{u}} \\ is^{-1}\,\tilde{\mathbf{t}} \end{bmatrix} \tag{12}$$

the above and eqn (8) imply that

$$\tilde{\mathbf{t}} = -is\,[\mathbf{B}\mathbf{E}_1\mathbf{\Xi} + \bar{\mathbf{B}}\mathbf{E}_2\mathbf{\Delta}] \tag{13}$$

where

$$\mathbf{B}(y) = [\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3]$$

It is obvious that both **A** and **B** are non-singular. In Appendix A, we prove that $\mathbf{BA}^{-1}$ was independent of y.

3. Statement of the problem and the singular integral equations

For each layer, the general solutions for displacements and stresses are expressed in terms of six unknown constants Ξ_i and Δ_i; $i = 1, 2, 3$. Consequently, complete solution for a N-layer plate requires a set of $6N$ appropriate boundary and interface conditions to determine $6N$ unknown constants.

The problem subjected to the following boundary and continuity conditions:

$$\begin{aligned} (\mathbf{t})_1^+ &= \mathbf{f}_u(x); \quad |x| < \infty \\ (\mathbf{t})_N^+ &= \mathbf{f}_l(x); \quad |x| < \infty \\ (\mathbf{u})_k^- &= (\mathbf{u})_{k+1}^+; \quad |x| < \infty, \quad k = 1, 2, \ldots, (j-1), (j+1), \ldots, (N-1) \\ (\mathbf{t})_k^- &= (t)_{k+1}^+; \quad |x| < \infty, \quad k = 1, 2, \ldots, (N-1) \end{aligned} \tag{14}$$

and

$$\begin{aligned} (\mathbf{u})_j^- &= (\mathbf{u})_{j+1}^+; \quad a < |x| < \infty \\ (\mathbf{t})_{j+1}^+ &= \mathbf{t}_0(x) = [t_{01} \quad t_{02} \quad t_{03}]^T; \quad |x| < a \\ \oint_C \mathrm{d}\mathbf{u} &= 0 \end{aligned} \tag{15}$$

where the subscripts k refer to the layer number, the superscripts + and − indicate the corresponding value evaluated at the upper/lower surface of the plate and $\mathbf{f}_u$, $\mathbf{f}_l$, $\mathbf{t}_0$ are the distribution functions for the applied traction on the upper, lower bounding surfaces and crack surface, respectively. C is any contour enclosing the crack and eqn $(15)_3$ represents the single value of displacements.

Introduce

$$\boldsymbol{\psi}(t) = \frac{\mathrm{d}}{\mathrm{d}x}[(\mathbf{u})_j - (\mathbf{u})_{j+1}] \quad |x| < \infty \tag{16}$$

Then, the displacement continuity condition outside the crack and the single-valuedness of displacements can be rewritten as

$$\boldsymbol{\psi}(x) = 0, \quad a < |x| < \infty, \quad \int_{-a}^{a} \boldsymbol{\psi}(x)\,\mathrm{d}x = 0 \tag{17}$$

So far, a system of $6N$ equations is yielded for the $6N$ unknown constants. In order to circumvent lengthy algebraic manipulations, the stiffness matrix formulation (Chatterjee, 1977) is extended to the analysis of current interface crack problems under a state of generalized plane deformation.

The traction vectors $\tilde{\mathbf{t}}^{\pm}(s)$ of the k-th layer can be expressed in terms of the corresponding displacement vectors $\tilde{\mathbf{u}}^{\pm}(s)$ as

$$\begin{bmatrix} \tilde{\mathbf{t}}^{+} \\ -\tilde{\mathbf{t}}^{-} \end{bmatrix}_k = \mathbf{k}_k \begin{bmatrix} \tilde{\mathbf{u}}^{+} \\ \tilde{\mathbf{u}}^{-} \end{bmatrix}_k \tag{18}$$

where $\mathbf{k}_k$ is the 6×6 local stiffness matrix of the k-th layer, which is given in Appendix B (which is Hermitian when p_i is distinct).

We also can rewrite eqn (18) as

$$\begin{bmatrix} \tilde{\mathbf{t}}^{+} \\ -\tilde{\mathbf{t}}^{-} \end{bmatrix}_k = \begin{bmatrix} \mathbf{k}_{11}^{k} & \mathbf{k}_{12}^{k} \\ \mathbf{k}_{21}^{k} & \mathbf{k}_{22}^{k} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{d}}_k^{+} \\ \tilde{\mathbf{d}}_k^{-} \end{bmatrix} \tag{19}$$

where, $\tilde{\mathbf{d}}_k^{\pm} = (\tilde{\mathbf{u}})_k^{\pm}$ and $\mathbf{k}_{ij}^{k}(s)$, $i, j = 1, 2$ are the 3×3 submatrices of $\mathbf{k}_k\,(s)$.

To be concise, we introduce

$$\tilde{\boldsymbol{\delta}}_1(s) = \tilde{\mathbf{d}}_1^{+}(s), \quad \tilde{\boldsymbol{\delta}}_{N+1}(s) = \tilde{\mathbf{d}}_N^{-}(s)$$

$$\tilde{\boldsymbol{\delta}}_{k+1}(s) = \tilde{\mathbf{d}}_k^{-}(s) = \tilde{\mathbf{d}}_{k+1}^{+}(s); \quad k \neq j$$

Then combining eqns (14) and (19) the traction continuity condition on the crack plane can be written as

$$\mathbf{k}_{21}^{j}\tilde{\boldsymbol{\delta}}_j + \mathbf{k}_{22}^{j}\tilde{\mathbf{d}}_j^{-} + \mathbf{k}_{11}^{j+1}\tilde{\mathbf{d}}_{j+1}^{+} + \mathbf{k}_{12}^{j+1}\tilde{\boldsymbol{\delta}}_{j+2} = 0 \tag{20}$$

From eqns (16) and (17), one can obtain

$$\Delta\tilde{\mathbf{d}}_{j+1} = \tilde{\mathbf{d}}_j^-(s) - \tilde{\mathbf{d}}_{j+1}^+(s) = \frac{i}{s}\int_{-a}^{a} \psi(x)\,\mathrm{e}^{isx}\,\mathrm{d}x \tag{21}$$

Let

$$\tilde{\mathbf{d}}_j^- = \tilde{\boldsymbol{\delta}}_{j+1} + \mathbf{C}_1\Delta\tilde{\mathbf{d}}_{j+1}, \quad \tilde{\mathbf{d}}_{j+1}^+ = \tilde{\boldsymbol{\delta}}_{j+1} + \mathbf{C}_2\Delta\tilde{\mathbf{d}}_{j+1} \tag{22}$$

where $\tilde{\boldsymbol{\delta}}_{j+1}$ is a vector with four components representing the interfacial displacements between the j-th and $(j+1)$-th layers without crack and $\mathbf{C}_i$; $i = 1, 2$ are the 3×3 unknown constant matrices yet to be evaluated with $\mathbf{C}_2 - \mathbf{C}_1 = \mathbf{I}$.

In Appendix B, we derived the asymptotic nature of the matrix $s^{-1}\mathbf{k}_k(s)$ as

$$\lim_{s\to+\infty}\frac{1}{s}\mathbf{k}_k(s) = \begin{bmatrix} \mathbf{Y}_k^{-1} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Y}}_k^{-1} \end{bmatrix}$$

$$\lim_{s\to-\infty}\frac{1}{s}\mathbf{k}_k(s) = \begin{bmatrix} -\bar{\mathbf{Y}}_k^{-1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{Y}_k^{-1} \end{bmatrix} \tag{23}$$

where $\mathbf{Y}_k = i\mathbf{A}\mathbf{B}^{-1}$, is Hermitian (Ting, 1988; Yang et al., 1997).

So, the asymptotic behaviors of $s^{-1}\mathbf{k}_k(s)$ and vector $\tilde{\boldsymbol{\delta}}_k(s)$ can be expressed as

$$\lim_{|s|\to\infty}\frac{1}{s}\begin{bmatrix} \mathbf{k}_{11}^k(s) & \mathbf{k}_{12}^k(s) \\ \mathbf{k}_{21}^k(s) & \mathbf{k}_{22}^k(s) \end{bmatrix} = \begin{bmatrix} \mathbf{k}_{11\infty}^k & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_{22\infty}^k \end{bmatrix} \lim_{|s|\to\infty}\tilde{\boldsymbol{\delta}}_k(s) = 0, \quad k = 1, 2, \ldots, (N+1) \tag{24}$$

Combining eqns (20) and (24), $\mathbf{C}_i$ are readily evaluated as

$$\mathbf{C}_1 = \mathbf{k}_\infty \mathbf{k}_{11\infty}^{j+1}, \quad \mathbf{C}_2 = -\mathbf{k}_\infty \mathbf{k}_{22\infty}^{j} \tag{25}$$

in which the matrix $\mathbf{k}_\infty$ is defined as

$$\mathbf{k}_\infty = [\mathbf{k}_{11\infty}^{j+1} + \mathbf{k}_{22\infty}^{j}]^{-1} \tag{26}$$

Through successive applications of eqns (14) to the layer stiffness matrix eqn (19), together with eqns (20), (22) and (25), a global stiffness matrix equation for an N-layer medium containing an interface crack between the j-th and $(j+1)$-th layers is constructed as

$$\begin{aligned}
&\mathbf{k}_{11}^1\tilde{\boldsymbol{\delta}}_1 + \mathbf{k}_{12}^1\tilde{\boldsymbol{\delta}}_2 = \tilde{\mathbf{f}}_\mathrm{u}\\
&\mathbf{k}_{21}^k\tilde{\boldsymbol{\delta}}_k + (\mathbf{k}_{22}^k + \mathbf{k}_{11}^{k+1})\tilde{\boldsymbol{\delta}}_{k+1} + \mathbf{k}_{12}^{k+1}\tilde{\boldsymbol{\delta}}_{k+2} = 0\\
&k = 1, 2, \ldots, (j-2), (j+2), \ldots, (N-1), \quad j \geqslant 3\\
&\mathbf{k}_{21}^{j-1}\tilde{\boldsymbol{\delta}}_{j-1} + (\mathbf{k}_{22}^{j-1} + \mathbf{k}_{11}^{j})\tilde{\boldsymbol{\delta}}_j + \mathbf{k}_{12}^j\tilde{\boldsymbol{\delta}}_{j+1} = -\mathbf{k}_{12}^j\mathbf{k}_\infty\mathbf{k}_{11\infty}^{j+1}\Delta\tilde{\mathbf{d}}_{j+1}\\
&\mathbf{k}_{21}^j\tilde{\boldsymbol{\delta}}_j + (\mathbf{k}_{22}^j + \mathbf{k}_{11}^{j+1})\tilde{\boldsymbol{\delta}}_{j+1} + \mathbf{k}_{12}^{j+1}\tilde{\boldsymbol{\delta}}_{j+1} = (\mathbf{k}_{11}^{j+1}\mathbf{k}_\infty\mathbf{k}_{22\infty}^j - \mathbf{k}_{22}^j\mathbf{k}_\infty\mathbf{k}_{11\infty}^{j+1})\Delta\tilde{\mathbf{d}}_{j+1}\\
&\mathbf{k}_{21}^{j+1}\tilde{\boldsymbol{\delta}}_{j+1} + (\mathbf{k}_{22}^{j+1} + \mathbf{k}_{11}^{j+2})\tilde{\boldsymbol{\delta}}_{j+2} + \mathbf{k}_{12}^{j+2}\tilde{\boldsymbol{\delta}}_{j+3} = -\mathbf{k}_{21}^{j+1}\mathbf{k}_\infty\mathbf{k}_{22\infty}^j\Delta\tilde{\mathbf{d}}_{j+1}\\
&\mathbf{k}_{21}^N\tilde{\boldsymbol{\delta}}_N + \mathbf{k}_{22}^N\tilde{\boldsymbol{\delta}}_{N+1} = -\tilde{\mathbf{f}}_\mathrm{l}
\end{aligned} \tag{27}$$

which can be expressed in matrix notation as

$$\mathbf{K}\tilde{\boldsymbol{\delta}} = \tilde{\mathbf{f}} + \mathbf{G}\Delta\tilde{\mathbf{d}} \tag{28}$$

where $\mathbf{K}(s)$ is the $3(N+1)\times 3(N+1)$ banded and Hermitian global stiffness matrix with half band width 6, $\tilde{\boldsymbol{\delta}}(s)$ is the vector of length $3(N+1)$-component column for the unknown interfacial displacements, $\tilde{\mathbf{f}}(s)$ is the vector of length $3(N+1)$-component column representing the prescribed traction applied on the upper and lower bounding surfaces, $\mathbf{G}(s)$ is the $3(N+1)\times 3(N+1)$ matrix containing the geometry and material properties of the j-th and $(j+1)$-ith layers, and $\Delta\tilde{\mathbf{d}}(s)$ is a $3(N+1)$-component column representing the crack surface displacements.

Combining eqns (19), (22), (25) and taking the inverse Fourier transform, a system of integral equations for ψ is obtained as

$$\mathbf{t}_{j+1}^{+} = \frac{i}{2\pi}\int_{-a}^{a}\left[\int_{-\infty}^{+\infty}\frac{1}{s}\mathbf{M}(s)\,\mathrm{e}^{is(t-x)}\,\mathrm{d}s\right]\boldsymbol{\psi}(t)\,\mathrm{d}t + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\boldsymbol{\mu}(s)\,\mathrm{e}^{isx}\,\mathrm{d}s, \quad |x|<\infty \tag{29}$$

where

$$\mathbf{M} = -\mathbf{k}_{11}^{j+1}\mathbf{k}_{\infty}\mathbf{k}_{22\infty}^{j} + \sum_{n=1}^{2}\mathbf{k}_{1n}^{j+1}\mathbf{H}_n$$

$$\boldsymbol{\mu}(s) = \sum_{n=1}^{2}\mathbf{k}_{1n}^{j+1}(\mathbf{L}_{(j+n)1}\tilde{\mathbf{f}}_{\mathrm{u}} - \mathbf{L}_{(j+n)(N+1)}\tilde{\mathbf{f}}_{1}) \tag{30}$$

with

$$\mathbf{H}_n = -\mathbf{L}_{(j+n)j}\mathbf{k}_{12}^{j}\mathbf{k}_{\infty}\mathbf{k}_{11\infty}^{j+1} + \mathbf{L}_{(j+n)(j+1)}(\mathbf{k}_{11}^{j+1}\mathbf{k}_{\infty}\mathbf{k}_{22\infty}^{j} - \mathbf{k}_{22}^{j}\mathbf{k}_{\infty}\mathbf{k}_{11\infty}^{j+1}) + \mathbf{L}_{(j+n)(j+2)}\mathbf{k}_{21}^{j+1}\mathbf{k}_{\infty}\mathbf{k}_{22\infty}^{j} \tag{31}$$

and $\mathbf{L}_{ij}(s)$ are the 3×3 submatrices of $\mathbf{L}(s) = \mathbf{K}^{-1}(s)$.

For the purpose of examining the singular behavior of the integral equations, the asymptotic behaviors of the matrix $s^{-1}\mathbf{M}(s)$ and of the forcing vector $\boldsymbol{\mu}(s)$ must first be investigated. From eqns (24) and (30), the corresponding asymptotic expressions are obtained as

$$\lim_{|s|\to\infty}\frac{1}{s}\mathbf{M}(s) = -\mathbf{k}_{11\infty}^{j+1}\mathbf{k}_{\infty}\mathbf{k}_{22\infty}^{j} = -\mathbf{M}_{\infty}$$

$$\lim_{|s|\to\infty}\boldsymbol{\mu}(s) = 0 \tag{32}$$

By employing eqn (23) as $s>0$, we have

$$\mathbf{M}_{+\infty}^{-1} = -(\mathbf{Y}_{j+1} + \bar{\mathbf{Y}}_{j}) \tag{33}$$

which means $\mathbf{M}_{+\infty}$ is Hermitian. To make further progress, write $\mathbf{M}_{+\infty}$ in its real and imaginary parts:

$$\mathbf{M}_{+\infty} = \mathbf{D}_{\infty} + i\mathbf{W}_{\infty} \tag{34}$$

where $\mathbf{D}_{\infty}$ is real and symmetric and $\mathbf{W}_{\infty}$ is real and antisymmetric.

Similarly, for $s>0$, from eqn (21) we can obtain

$$\mathbf{M}_{-\infty}^{-1} = -(\bar{\mathbf{Y}}_{j+1}+\mathbf{Y}_j) \tag{35}$$

and

$$\mathbf{M}_{-\infty} = -\mathbf{D}_\infty + i\mathbf{W}_\infty \tag{36}$$

From eqns (34) and (36) it follows that

$$\mathbf{M}_\infty = \frac{s}{|s|}\mathbf{D}_\infty + i\mathbf{W}_\infty \tag{37}$$

Obviously, the singularities of the integral equations are attributable to the asymptotic values of the matrix $s^{-1}\mathbf{M}(s)$ as $s \to \infty$. By separating the singular part $\mathbf{M}$ and using the integral formulas (Friedman, 1969)

$$\int_{-\infty}^{+\infty} \frac{s}{|s|} e^{is(t-x)}\,ds = \frac{2i}{t-x} \quad \text{and} \quad \int_{-\infty}^{+\infty} e^{is(t-x)}\,ds = 2\pi\delta(t-x) \tag{38}$$

eqns (29) may be expressed as

$$-\frac{i}{2\pi}\int_{-a}^{a}\left[\int_{-\infty}^{+\infty}\mathbf{M}_\infty e^{-is(t-x)}\,ds\right]\boldsymbol{\psi}(t)\,dt + \frac{i}{2\pi}\int_{-a}^{a}\left[\int_{-\infty}^{+\infty}\left(\frac{1}{s}\mathbf{M}(s)+\mathbf{M}_\infty\right)e^{is(t-x)}\,ds\right]\boldsymbol{\psi}(t)\,dt$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\boldsymbol{\mu}(s)\,e^{-isx}\,ds = \mathbf{t}_0 \quad |x|<a \tag{39}$$

The above is rearranged in the form

$$\mathbf{D}_\infty^{-1}\mathbf{W}_\infty\boldsymbol{\psi}(x) + \frac{1}{\pi}\int_{-a}^{a}\frac{\boldsymbol{\psi}(t)}{t-x}\,dt + \mathbf{D}_\infty^{-1}\frac{i}{2\pi}\int_{-a}^{a}\left[\int_{-\infty}^{+\infty}\left(\frac{1}{s}\mathbf{M}(s)+\mathbf{M}_\infty\right)e^{is(t-x)}\,ds\right]\boldsymbol{\psi}(t)\,dt$$
$$+\frac{\mathbf{D}_\infty^{-1}}{2\pi}\int_{-\infty}^{+\infty}\boldsymbol{\mu}(s)\,e^{-isx}\,ds = \mathbf{D}_\infty^{-1}\mathbf{t}_0 \quad |x|<a \tag{40}$$

There exists a matrix $\boldsymbol{\Lambda}^{-1}$ which is composed of eigenvectors of $\mathbf{D}_\infty^{-1}\mathbf{W}_\infty$ to make $\mathbf{D}_\infty^{-1}\mathbf{W}_\infty$ diagonal, i.e.

$$\boldsymbol{\Lambda}\mathbf{D}_\infty^{-1}\mathbf{W}_\infty\boldsymbol{\Lambda}^{-1} = \mathrm{diag}\,[\lambda_i] \tag{41}$$

where λ_i are the eigenvalues of $\mathbf{D}_\infty^{-1}\mathbf{W}_\infty$. Therefore, eqn (40) can be written as

$$\mathrm{diag}\,[\lambda_i]\boldsymbol{\Lambda}\boldsymbol{\psi}(x) + \frac{\boldsymbol{\Lambda}}{\pi}\int_{-a}^{a}\frac{\boldsymbol{\psi}(t)}{t-x}\,dt + \boldsymbol{\Lambda}\mathbf{D}_\infty^{-1}\frac{i}{2\pi}\int_{-a}^{a}\left[\int_{-\infty}^{+\infty}\left(\frac{1}{s}\mathbf{M}(s)+\mathbf{M}_\infty\right)e^{is(t-x)}\,ds\right]\boldsymbol{\psi}(t)\,dt$$
$$+\frac{\boldsymbol{\Lambda}\mathbf{D}_\infty^{-1}}{2\pi}\int_{-\infty}^{+\infty}\boldsymbol{\mu}(s)\,e^{-isx}\,ds = \boldsymbol{\Lambda}\mathbf{D}_\infty^{-1}\mathbf{t}_0 \quad |x|<a \tag{42}$$

Let

$$\mathbf{\Lambda}\boldsymbol{\psi}(x) = [\boldsymbol{\psi}_{\Lambda 1} \quad \boldsymbol{\psi}_{\Lambda 2} \quad \boldsymbol{\psi}_{\Lambda 3} \quad \boldsymbol{\psi}_{\Lambda 4}]^T = \boldsymbol{\psi}_{\Lambda}(x)$$

$$\mathbf{\Lambda}\mathbf{D}_{\infty}^{-1}\boldsymbol{\mu}(s) = \boldsymbol{\upsilon}(s)$$

$$\mathbf{\Lambda}\mathbf{D}_{\infty}^{-1}\mathbf{t}_0(s) = \mathbf{P}(s)$$

Thus, eqns (40) can be decoupled as

$$\lambda_i\psi_{\Lambda i}(x) + \frac{1}{\pi}\int_{-a}^{a}\frac{\psi_{\Lambda i}(t)}{t-x}\,\mathrm{d}t + \int_{-a}^{a}\sum_{n=1}^{4}F_{in}\psi_{\Lambda i}(t)\,\mathrm{d}t + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\upsilon_i(s)\,\mathrm{e}^{-isx}\,\mathrm{d}s = P_i(x) \quad |x| < a, \quad i = 1,2,3 \tag{43}$$

where $[F_{ij}]$ whose elements are the bounded Fredholm kernels, is a 3×3 matrix defined as

$$[F_{ij}] = \mathbf{\Lambda}\mathbf{D}_{\infty}^{-1}\frac{i}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{1}{s}\mathbf{M}(s) + \mathbf{M}_{\infty}\right)\mathrm{e}^{is(t-x)}\,\mathrm{d}s\mathbf{\Lambda}^{-1} \tag{44}$$

The analytic solution of eqn (43) has been extensively studied (Muskhelishvilli, 1953) by using the regularization method, which is this case becomes cumbersome. Here we try to use an approximation method described by Erdogan (1969) to find the stress intensity factors.

In the normalized interval of $\xi = x/a$ and $\tau = t/a$, it can be shown that the fundamental functions of dominant part of the integral equations are the weight function of the Jacobi polynomials. In order to preserve the correct nature of singularities of the problem, the solution of eqn (43) may be expressed as (Erdogan, 1984).

$$\psi_{\Lambda i}(t) = g_i(t)w_i(t), \quad g_i(t) = \sum_{n=0}^{\infty}C_n^i P_n^{(\alpha_i\beta_i)}(t) \quad |t| < 1 \tag{45}$$

where

$$w_i(\tau) = (1-\tau)^{\alpha_i}(1+\tau)^{\beta_i}$$

$$\alpha_k = \frac{i}{2\pi i}\ln\frac{1-\lambda_k i}{1+\lambda_k i} - \frac{1}{2}, \quad \beta_k = -\frac{i}{2\pi i}\ln\frac{1-\lambda_k i}{1+\lambda_k i} - \frac{1}{2} \quad (k = 1,2)$$

$$\alpha_3 = \beta_3 = -\frac{1}{2} \tag{46}$$

and $P_n^{(\alpha_i,\beta_i)}(t)$ is the Jacobi polynomial, C_n^k are the unknown constants yet to be evaluated.

By considering the orthogonality relations of Jacobi polynomials (Gradshteyn and Ryzhik, 1980)

$$\int_{-1}^{1}P_n^{(\alpha,\beta)}(t)P_k^{(\alpha,\beta)}(t)w(t)\,\mathrm{d}t = \begin{cases}0, & n \neq k\\ \theta_k^{(\alpha,\beta)}, & n = k\end{cases}$$

$$\theta_0^{(\alpha,\beta)} = \frac{2^{(\alpha+\beta+1)}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

$$\theta_k^{(\alpha,\beta)} = \frac{2^{(\alpha+\beta+1)}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)} \quad k = 1, 2, \ldots \tag{47}$$

together with $P_0^{(\alpha,\beta)}(t) = 1$ it can be concluded that the condition (17) is identically satisfied provided that $C_0^k = 0$.

Using the following relation (Karpenko, 1966):

$$\lambda_k P_n^{(\alpha_k,\beta_k)}(x) w_k(x) + \frac{1}{\pi}\int_{-1}^{1} P_n^{(\alpha_k,\beta_k)}(t)\frac{w_k(t)}{t-x}\,\mathrm{d}t$$
$$= \begin{cases} \dfrac{(1+\lambda_k^2)^{1/2}}{2} P_{n-1}^{(-\alpha_k,-\beta_k)}(x) & |x| < 1 \\ \dfrac{(1+\lambda_k^2)^{1/2}}{2}[(x-1)^{\alpha_k}(x-1)^{\beta_k} P_n^{(\alpha_k,\beta_k)}(x) + G_{kn}^{\infty}(x)] & |x| > 1 \end{cases} \tag{48}$$

where $G_{kn}^{\infty}(x)$ is the principal part of $w_k(x)P_n^{(\alpha_k,\beta_k)}(x)$ at infinity, the singularity of eqn (43) in removed such that

$$\sum_{n=1}^{\infty} C_n^k \left[\frac{(1+\lambda_k^2)^{1/2}}{2} P_{n-1}^{(-\alpha_k,-\beta_k)}(\zeta)\right] + \sum_{n=1}^{\infty}\sum_{m=1}^{4} H_n^{km}(\zeta) C_n^m = e_k(\zeta) \quad k = 1, 2, 3; \quad |\zeta| < 1 \tag{49}$$

where

$$H_n^{km}(\zeta) = a\int_{-1}^{1} F_{lm}(\zeta,\tau) P_n^{(\alpha_l,\beta_l)}(\tau) w_l(\tau)\,\mathrm{d}\tau$$

$$\mathbf{e}(\zeta) = [e_1(\zeta) \quad e_2(\zeta) \quad e_3(\zeta) \quad e_4(\zeta)]^T = \mathbf{P}(\zeta) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boldsymbol{\upsilon}(s)\,\mathrm{e}^{-isa\zeta}\,\mathrm{d}s$$

By using the orthogonality relations (47) again the following algebraic equations for C_n^k are obtained as

$$\frac{(1+\lambda_l^2)^{1/2}}{2}\theta_{k-1}^{(-\alpha_l,-\beta_l)} C_k^l + \sum_{n=1}^{\infty}\sum_{m=1}^{4} Y_{kn}^{lm} C_n^m = q_{kl} \quad l = 1, 2, 3; \quad k = 1, 2, \ldots \tag{50}$$

where

$$q_{kl} = \int_{-1}^{1} e_1(\zeta) P_{k-1}^{(-\alpha_l,-\beta_l)}(\zeta) w_l(\zeta)\,\mathrm{d}\zeta$$

$$Y_{kn}^{lm} = \int_{-1}^{1} H_n^{lm} P_{k-1}^{(-\alpha_l,-\beta_l)}(\zeta) w_l(\zeta)\,\mathrm{d}\zeta \tag{51}$$

In fact, the series in eqn (50) is truncated after the first N terms leading to a system of $3N$ simultaneous linear algebraic equations for the constants C_n^k. The number N must be large enough to yield the results that are within a specified degree of accuracy. After the constants C_n^k evaluated from eqn (50), the traction components σ_{22}, σ_{12}, σ_{32} for $|\zeta| > 1$ on the plane containing the crack can be obtained in terms of the solutions of eqns (43) as

$$\mathbf{t}(\zeta) = \mathbf{D}_\infty \mathbf{\Lambda}^{-1} \sum_{n=1}^{N} \begin{Bmatrix} \dfrac{(1+\lambda_1^2)^{1/2}}{2}(\zeta-1)^{\alpha_1}(\zeta+1)^{\beta_l} P_n^{(\alpha_1,\beta_1)}(\zeta) C_n^1 \\ \dfrac{(1+\lambda_2^2)^{1/2}}{2}(\zeta-1)^{\alpha_2}(\zeta+1)^{\beta_2} P_n^{(\alpha_2,\beta_2)}(\zeta) C_n^2 \\ \dfrac{(1+\lambda_3^2)^{1/2}}{2}(\zeta-1)^{\alpha_3}(\zeta+1)^{\beta_{31}} P_n^{(\alpha_3,\beta_3)}(\zeta) C_n^3 \end{Bmatrix} + \mathbf{O}(1) \tag{52}$$

where $\mathbf{O}(\cdot)$ represents the higher-order terms. The stress intensity factor can be obtained as (at the right-hand-side crack tip).

$$\begin{bmatrix} k_{\mathrm{I}} \\ k_{\mathrm{II}} \\ k^{\mathrm{III}} \end{bmatrix} = \lim_{\zeta\to 1} \mathbf{D}_\infty \mathbf{\Lambda}^{-1} \sum_{n=1}^{N} \begin{Bmatrix} \dfrac{(1+\lambda_1^2)^{1/2}}{2}(\zeta-1)^{\alpha_1}(\zeta+1)^{\beta_1} P^{(\alpha_1,\beta_1)}(\zeta) C_n^1 \\ \vdots \\ \vdots \\ \dfrac{(1+\lambda_3^2)^{1/2}}{2}(\zeta-1)^{\alpha_3}(\zeta+1)^{\beta_3} P_n^{(\alpha_3,\beta_3)}(\zeta) C_n^3 \end{Bmatrix}$$

$$= \mathbf{D}_\infty \Lambda^{-1} \sum_{n=1}^{N} \begin{Bmatrix} \dfrac{(1+\lambda_1^2)^{1/2}}{2} 2^{\beta_1} P_n^{(\alpha_1,\beta_1)}(1) C_n^1 \\ \vdots \\ \vdots \\ \dfrac{(1+\lambda_3^2)^{1/2}}{2} 2^{\beta_3} P_n^{(\alpha_3,\beta_3)}(1) C_n^3 \end{Bmatrix} \tag{53}$$

where we have used the fact that $G_{kn}^{\infty}(x)$, the princikpal part of $w_k(\zeta) P_n^{(\alpha_k,\beta_k)}(\zeta)$ is bounded and the Jacobi polynomials $P_n^{(\alpha_k,\beta_k)}(\zeta)$ may be calculated by the following (see e.g. Gradshteyn and Ryzhik (1980)):

$$P_n^{(\alpha,\beta)}(\zeta) = \frac{(-1)^n}{2^n n!}(1-\zeta)^{-\alpha}(1+\zeta)^{-\beta} \frac{\mathrm{d}^n}{\mathrm{d}x^n}[(1-\zeta)^{\alpha+n}(1+\zeta)^{\beta+n}]$$

4. Numerical result

In this section, numerical examples are given to illustrate the method of solution outlined in Section 3. The material (PVDF) parameters used in the numerical computation are given in Appendix C. The laminated plate is balanced symmetrically which has ply angles [0°/45°/90°]s. All the layers are considered to have the same thickness h. The Jacobi-type integrals and the improper

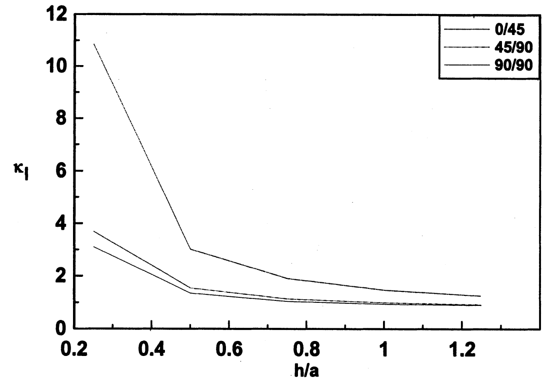

Fig. 2. Variation of $\kappa_{\rm I}$ under mode I loading as a function of h/a for various crack locations.

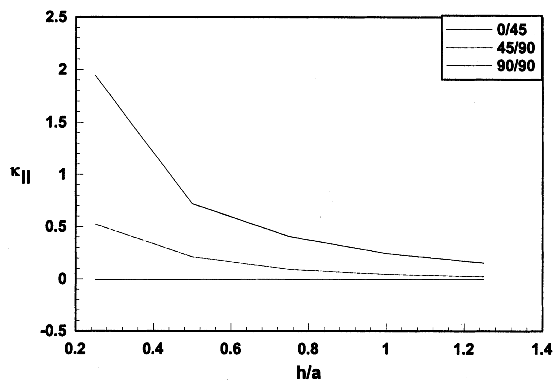

Fig. 3. Variation of $\kappa_{\rm II}$ under mode I loading as a function of h/a for various crack locations.

integrals, were numerically evaluated by employing the Gauss–Jacobi and Gauss–Legendre quadrature formulae (Davis and Rabinowitz, 1984), respectively.

It is noted that the intensity factors presented in this study are evaluated at the right-hand side crack tip. Numerical, value of the intensity factors have been calculated for the following three modes: in-plane normal (mode I), in-plane shear (mode II), anti-plane (mode III) crack surface loadings, which are normalized by the corresponding $\sigma\sqrt{\pi a}$.

4.1. Mode I loading

Under mode I loading, the major stress intensity factor $\kappa_{\rm I}$ and coupling stress intensity factors $\kappa_{\rm II}$ against h/a for various crack locations are plotted in Fig. 2 and Fig. 3, respectively.

4.2. Mode II loading

Under mode II loading, the major stress intensity factor $\kappa_{\rm II}$ and coupling stress intensity factors $\kappa_{\rm I}$ against h/a for various crack locations are displayed in Figs 4 and 5, respectively.

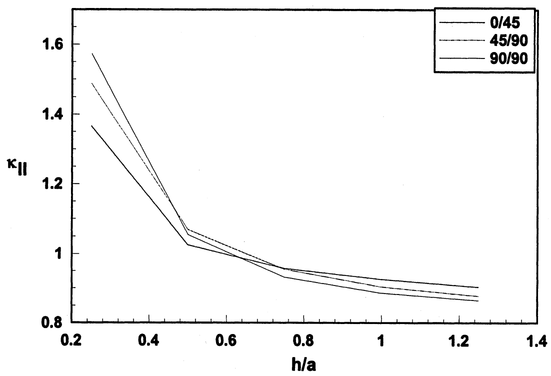

Fig. 4. Variation of κ_{II} under mode II loading as a function of h/a for various crack locations.

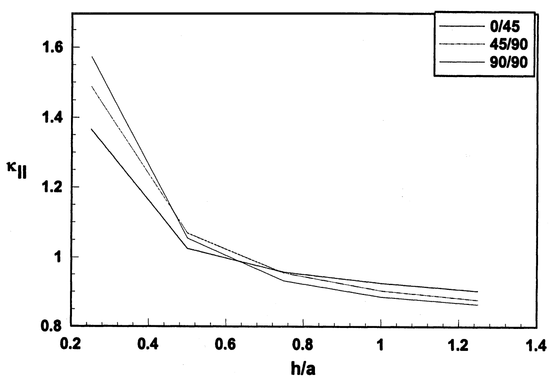

Fig. 5. Variation of κ_{I} under mode II loading as a function of h/a for various crack locations.

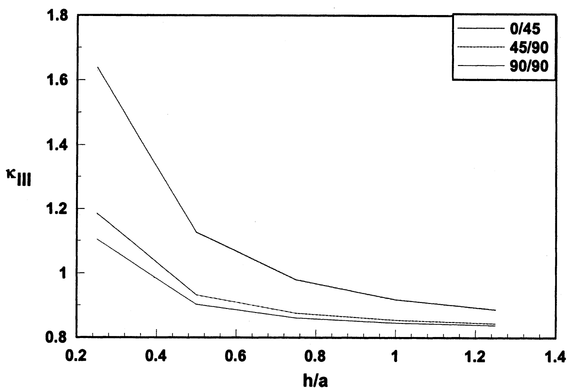

Fig. 6. Variation of κ_{III} under mode III loading as a function of h/a for various crack locations.

4.3. Mode III loading

Under mode III loading, the major stress intensity factor $\kappa_{\rm III}$ against h/a for various crack locations are shown in Fig. 6.

Appendix A

In this Appendix, we will prove that $\mathbf{BA}^{-1}$ is independent of y.

For example, let p_1 be a double root, then we have

$$\mathbf{A}^{(1)} = [\mathbf{a}_1 \quad \mathbf{a}' \quad \mathbf{a}_3] \quad \mathbf{B}^{(1)} = [\mathbf{b}_1 \quad \mathbf{b}' \quad \mathbf{b}_3] \tag{A1}$$

In the literature (Yang et al., 1997), it has been proved that matrix $i\mathbf{B}^{(1)}(\mathbf{A}^{(1)})^{-1}$ is a Hermitian matrix, i.e.

$$[\mathbf{b}_1 \quad \mathbf{b}_1' \quad \mathbf{b}_3]\cdot[\mathbf{a}_1 \quad \mathbf{a}_1' \quad \mathbf{a}_3]^{-1} = -\begin{bmatrix}\bar{\mathbf{a}}_1\\ \bar{\mathbf{a}}_1'\\ \bar{\mathbf{a}}_3\end{bmatrix}^{-1}\cdot\begin{bmatrix}\bar{\mathbf{b}}_1\\ \bar{\mathbf{b}}_1'\\ \bar{\mathbf{b}}_3\end{bmatrix} \tag{A2}$$

In this case,

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_1' + y\mathbf{a}_1 \quad \mathbf{a}_3] \quad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_1' + y\mathbf{b}_1 \quad \mathbf{b}_3] \tag{A3}$$

From eqn (A2), one can obtain

$$\begin{bmatrix}\bar{\mathbf{a}}_1\\ \bar{\mathbf{a}}_1'\\ \bar{\mathbf{a}}_3\end{bmatrix}[\mathbf{b}_1 \quad \mathbf{b}_1' \quad \mathbf{b}_3] = -\begin{bmatrix}\bar{\mathbf{b}}_1\\ \bar{\mathbf{b}}'\\ \bar{\mathbf{b}}_3\end{bmatrix}[\mathbf{a}_1 \quad \mathbf{a}_1' \quad \mathbf{a}_3] \tag{A4}$$

Thus, on can readily get

$$\begin{bmatrix}\bar{\mathbf{a}}_1\\ \bar{\mathbf{a}}_1'\\ \bar{\mathbf{a}}_3\end{bmatrix}[\mathbf{b}_1 \quad \mathbf{b}_1' + y\mathbf{b}_1 \quad \mathbf{b}_3] = -\begin{bmatrix}\bar{\mathbf{b}}_1\\ \bar{\mathbf{b}}'\\ \bar{\mathbf{b}}_3\end{bmatrix}[\mathbf{a}_1 \quad \mathbf{a}_1' + y\mathbf{a}_1 \quad \mathbf{a}_3] \tag{A5}$$

So

$$[\mathbf{b}_1 \quad \mathbf{b}_1' + y\mathbf{b}_1 \quad \mathbf{b}_3]\cdot[\mathbf{a}_1 \quad \mathbf{a}_1' + y\mathbf{a}_1 \quad \mathbf{a}_3]^{-1} = -\begin{bmatrix}\bar{\mathbf{a}}_1\\ \bar{\mathbf{a}}_1'\\ \bar{\mathbf{a}}_3\end{bmatrix}^{-1}\cdot\begin{bmatrix}\bar{\mathbf{b}}_1\\ \bar{\mathbf{b}}_1'\\ \bar{\mathbf{b}}_3\end{bmatrix} = [\mathbf{b}_1 \quad \mathbf{b}_1' \quad \mathbf{b}_3]\cdot[\mathbf{a}_1 \quad \mathbf{a}_1' \quad \mathbf{a}_3]^{-1}$$

which states that $\mathbf{BA}^{-1}$ is independent of y and is also a Hermitian matrix.

Analogously, the case that p_i is a k-fold root can be treated.

Appendix B

In this Appendix, we discuss the asymptotic behavior of the matrix $s^{-1}\mathbf{k}_k$.

By using eqns (9) and (13), for any layer (such as the k-th layer), we have,

$$\begin{bmatrix} \tilde{\mathbf{u}}^+ \\ \tilde{\mathbf{u}}^- \end{bmatrix} = \begin{bmatrix} \mathbf{A}^+\mathbf{E}_1^+ & \bar{\mathbf{A}}^+\mathbf{E}_2^+ \\ \mathbf{A}^-\mathbf{E}_1^- & \bar{\mathbf{A}}^-\mathbf{E}_2^- \end{bmatrix} \begin{bmatrix} \mathbf{\Xi} \\ \mathbf{\Delta} \end{bmatrix} \tag{B1}$$

and

$$\begin{bmatrix} \tilde{\mathbf{t}}^+ \\ \tilde{\mathbf{t}}^- \end{bmatrix} = -is \begin{bmatrix} \mathbf{B}^+\mathbf{E}_1^+ & \bar{\mathbf{B}}^+\mathbf{E}_2^+ \\ -\mathbf{B}^-\mathbf{E}_1^- & -\bar{\mathbf{B}}^-\mathbf{E}_2^- \end{bmatrix} \begin{bmatrix} \mathbf{\Xi} \\ \mathbf{\Delta} \end{bmatrix} \tag{B2}$$

where the superscripts + and − indicate the corresponding value evaluated at the upper/lower surface of the layer.

From eqns (B1) and (B2), it follows that

$$\begin{bmatrix} \tilde{\mathbf{t}}^+ \\ \tilde{\mathbf{t}}^- \end{bmatrix} = -is \begin{bmatrix} \mathbf{B}^+\mathbf{E}_1^+ & \bar{\mathbf{B}}^+\mathbf{E}_2^+ \\ -\mathbf{B}^-\mathbf{E}_1^- & -\bar{\mathbf{B}}^-\mathbf{E}_2^- \end{bmatrix} \begin{bmatrix} \mathbf{A}^+\mathbf{E}_1^+ & \bar{\mathbf{A}}^+\mathbf{E}_2^+ \\ \mathbf{A}^-\mathbf{E}_1^- & \bar{\mathbf{A}}^-\mathbf{E}_2^- \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{u}}^+ \\ \tilde{\mathbf{u}}^- \end{bmatrix} \tag{B3}$$

Thus, we have

$$\begin{aligned} \frac{1}{s}\mathbf{k}_k &= -i \begin{bmatrix} \mathbf{B}^+\mathbf{E}_1^+ & \bar{\mathbf{B}}^+\mathbf{E}_2^+ \\ -\mathbf{B}^-\mathbf{E}_1^- & -\bar{\mathbf{B}}^-\mathbf{E}_2^- \end{bmatrix} \begin{bmatrix} \mathbf{A}^+\mathbf{E}_1^+ & \bar{\mathbf{A}}^+\mathbf{E}_2^+ \\ \mathbf{A}^-\mathbf{E}_1^- & \bar{\mathbf{A}}^-\mathbf{E}_2^- \end{bmatrix}^{-1} \\ &= -i \begin{bmatrix} \mathbf{B}^+\mathbf{E}_1^+ & \bar{\mathbf{B}}^+\mathbf{E}_2^+ \\ -\mathbf{B}^-\mathbf{E}_1^- & -\bar{\mathbf{B}}^-\mathbf{E}_2^- \end{bmatrix} \begin{bmatrix} \bar{\mathbf{E}}_2^+ & \\ & \bar{\mathbf{E}}_1^- \end{bmatrix} \begin{bmatrix} \bar{\mathbf{E}}_2^+ & \\ & \bar{\mathbf{E}}_1^- \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}^+\mathbf{E}_1^+ & \bar{\mathbf{A}}^+\mathbf{E}_2^+ \\ \mathbf{A}^-\mathbf{E}_1^- & \bar{\mathbf{A}}^-\mathbf{E}_2^- \end{bmatrix}^{-1} \\ &= -i \begin{bmatrix} \mathbf{B}^+ & \bar{\mathbf{B}}^+\mathbf{E}_2^+\bar{\mathbf{E}}_1^- \\ -\mathbf{B}^-\mathbf{E}_1^-\bar{\mathbf{E}}_2^+ & -\bar{\mathbf{B}}^- \end{bmatrix} \begin{bmatrix} \mathbf{A}^+ & \bar{\mathbf{A}}^+\mathbf{E}_2^+\bar{\mathbf{E}}_1^- \\ \mathbf{A}^-\mathbf{E}_1^-\bar{\mathbf{E}}_2^+ & \bar{\mathbf{A}}^- \end{bmatrix}^{-1} \end{aligned} \tag{B4}$$

During the derivation of eqn (B4) we used the following relations

$$\bar{\mathbf{E}}_2^+\mathbf{E}_1^+ = \mathbf{I}, \quad \bar{\mathbf{E}}_2^-\mathbf{E}_1^- = \mathbf{I} \tag{B5}$$

which can proved by the definition of $\mathbf{E}_1$ and $\mathbf{E}_2$. Also, it is easy to obtain

$$\lim_{s\to+\infty} \mathbf{E}_1^-\bar{\mathbf{E}}_2^+ = 0 = \lim_{s\to+\infty} \mathbf{E}_2^+\bar{\mathbf{E}}_1^- \tag{B6}$$

We can write

$$\begin{bmatrix} \mathbf{A}^+ & \bar{\mathbf{A}}^+\mathbf{E}_2^+\bar{\mathbf{E}}_1^- \\ \mathbf{A}^-\mathbf{E}_1^-\bar{\mathbf{E}}_2^+ & \bar{\mathbf{A}}^- \end{bmatrix}^{-1}$$

explicitly as

$$\begin{bmatrix} \mathbf{A}^{+} & \bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-} \\ \mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+} & \bar{\mathbf{A}}^{-} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \tag{B7}$$

where

$$\mathbf{C}_{11} = -(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}[\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-} - \mathbf{A}^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}]^{-1}$$

$$\mathbf{C}_{12} = -(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}[\bar{\mathbf{A}}^{-} - \mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}]^{-1}$$

$$\mathbf{C}_{21} = [\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-} - \mathbf{A}^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}]^{-1}$$

$$\mathbf{C}_{22} = [\bar{\mathbf{A}}^{-} - \mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}]^{-1}$$

So, it follows that

$$\left(\frac{-1}{is}\mathbf{K}_k\right)_{11} = -\mathbf{B}^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}[\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-} - \mathbf{A}^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}]^{-1} + \bar{\mathbf{B}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}[\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-} - \mathbf{A}^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}]^{-1}$$

$$\left(\frac{-1}{is}\mathbf{K}_k\right)_{12} = -\mathbf{B}^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}[\bar{\mathbf{A}}^{-} - \mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}]^{-1} + \bar{\mathbf{B}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}[\bar{\mathbf{A}}^{-} - \mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}]^{-1}$$

$$\left(\frac{-1}{is}\mathbf{K}_k\right)_{21} = \mathbf{B}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}[\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-} - \mathbf{A}^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}]^{-1} - \bar{\mathbf{B}}^{-}[\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-} - \mathbf{A}^{+}(\mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+})^{-1}\bar{\mathbf{A}}^{-}]^{-1}$$

$$\left(\frac{-1}{is}\mathbf{K}_k\right)_{22} = \mathbf{B}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}[\bar{\mathbf{A}}^{-} - \mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}]^{-1} - \bar{\mathbf{B}}^{-}[\bar{\mathbf{A}}^{-} - \mathbf{A}^{-}\mathbf{E}_1^{-}\bar{\mathbf{E}}_2^{+}(\mathbf{A}^{+})^{-1}\bar{\mathbf{A}}^{+}\mathbf{E}_2^{+}\bar{\mathbf{E}}_1^{-}]^{-1}$$

By means of eqn (B6) and the conclusion of Appendix A, we can obtain

$$\lim_{s\to+\infty}\frac{1}{s}\mathbf{k}_k = -\begin{bmatrix} i\mathbf{B}^{+}(\mathbf{A}^{+})^{-1} & 0 \\ 0 & -i\bar{\mathbf{B}}^{-}(\bar{\mathbf{A}}^{-})^{-1} \end{bmatrix} = \begin{bmatrix} -i\mathbf{B}\mathbf{A}^{-1} & \\ & \overline{-i\mathbf{B}\mathbf{A}^{-1}} \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_k^{-1} & \\ & \bar{\mathbf{Y}}_k^{-1} \end{bmatrix} \tag{B8}$$

where $\mathbf{Y}_k = i\mathbf{A}\mathbf{B}^{-1}$, which is Hermitian.

For $s \to -\infty$ substituting

$$\begin{bmatrix} \bar{\mathbf{E}}_2^{+} & \\ & \bar{\mathbf{E}}_1^{-} \end{bmatrix} \quad \text{with} \begin{bmatrix} & \bar{\mathbf{E}}_2^{-} \\ \bar{\mathbf{E}}_1^{+} & \end{bmatrix}$$

in eqn (B4) we can readily obtain

$$\lim_{s\to+\infty}\frac{1}{s}\mathbf{k}_k=\begin{bmatrix}-i\overline{\mathbf{BA}^{-1}} & \\ & i\mathbf{BA}^{-1}\end{bmatrix}=\begin{bmatrix}-\bar{\mathbf{Y}}_k^{-1} & \\ & -\mathbf{Y}_k^{-1}\end{bmatrix} \tag{B9}$$

This completes the proof of eqn (23).

Appendix C

The material parameters of PVDF (Varadan et al., 1989):

$$\mathbf{c}^E=\begin{bmatrix}3.61 & 1.61 & 1.42 & 0 & 0 & 0\\ & 3.13 & 1.31 & 0 & 0 & 0\\ & & 1.63 & 0 & 0 & 0\\ & & & 0.55 & 0 & 0\\ & & & & 0.59 & 0\\ & & & & & 0.69\end{bmatrix}\ \mathrm{GPa}$$

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